The State Dependence Calculation of Three-body Cluster Energy for Nuclear Matter

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Abstract

It is shown that the method of lowest order constrained variational (LOCV) which is based on the cluster expansion theory is a reliable many-body technique to calculate the nuclear matter equation of state. In this respect, the state dependent correlation functions and the effective interactions which have been produced by the LOCV calculation with the Reid soft core and the Δ-Reid interactions are used to estimate the size of higher order cluster terms such as the effect of three-body cluster energy on nuclear matter. We find that the three-body cluster energy is less than 1 MeV beyond the nuclear matter saturation density and it has weaker density dependence than our previous calculation with the state-averaged correlation function and the effective interaction. Finally we conclude that the LOCV method is good enough to calculate other properties of quantal fluids.

I. INTRODUCTION

The method of lowest constrained variational (LOCV) was developed in 1977-1979 [1,2] to calculate the bulk properties of homogeneous nuclear fluid such as the saturation energy and density, the surface energy and the asymmetrical coefficient in the semi-empirical formula, by using the realistic nucleon-nucleon interactions i.e. the Reid [3] and the Δ-Reid [2] potentials. The LOCV method was successfully extended further for finite temperature calculation [4] and various thermodynamics properties of nuclear matter were evaluated. Recently this approach was reformulated to include more sophisticated interactions such as the V_{14} [5], the AV_{14} and the new Argonne AV_{18} [6] potentials. In these calculations [5,6] it was shown that the new AV_{18} potential, like the other N-N interactions, dose over bind nuclear matter and a very good agreement was found between the LOCV results and more complicated approaches such as the Bruckner-Hartree- Fock (BHF), the Correlated Basis Function (CBF), the Bruckner-Bethe (BB) and the Variational Hypernetted Chain (VHC) techniques. The results of these calculations are demonstrated in table 1 by presenting the saturation energy, the saturation density and the incompressibility of nuclear matter for different potentials and different many-body methods from reference [6]. According to this table, it is seen that only the results of the AV_{14} and the Δ-Reid potentials are close to the empirical values and on average the inclusion of density dependence three-nucleon interaction (TNI) improves the calculations.

In order to test the convergence of our LOCV results for nuclear matter, we performed calculations beyond the lowest order (the two-body cluster term) and the three-body cluster energy was calculated with the state-averaged correlation function [10] which in turn was generated by using the state-dependent LOCV correlation functions. The smallness of the normalization parameter (the convergence parameter) and the three-body cluster energy where indicated that at least up to the twice nuclear matter density, our expansion converges reasonably and it is a good approximation to stop after the two-body cluster terms [2,5,10]. LOCV formalism has several advantages with respect to the other many-body techniques like

the Brueckner-Bethe (BB) [7], the Variational Hypernetted Chain (VHC) [8], the Correlated Basis Function (CBF) [8] and the Brueckner-Hartree-Fock (BHF) [9] which go beyond the lowest order:

1) There is no free parameter in the LOCV method beside the N-N potentials i.e. it is fully self-consistent.

2) It considers constraint in the form normalization condition [11] to keep the higher cluster terms as small as possible. As we pointed above, this has been tested by calculating the three-body cluster terms with the state-averaged correlation functions.

3) It assumes a particular form for the long-range part of the correlation functions in order to perform an exact functional minimization of the two-body energy with respect to the short-range behavior of the correlation functions. 4) It dose functional minimization, rather than simply parameterizing the short-range behavior of the correlation functions. So in this respect it also saves an enormous computational time.

The aim of present work is to calculate the three-body cluster terms with the state-dependent correlations i.e. without making the state-averaged approximation. In this work we use the Reid soft core [3] and the Δ-Reid [2] potentials, since we are only interested in the size of three-body cluster energy. Beside this the results of our previous calculation shows that the Δ-Reid interaction can reasonably reproduce the properties of nuclear matter. On the other hand the Δ state is the most important configuration which modifies the nuclear forces and one may consider it as the origin of understanding of the three-body forces [12].

So the paper is organized as follows. We begin section II by describing briefly the LOCV formalism. Section III-A is devoted to the three-body cluster energy and the definition of the state-averaged correlation function and the effective interaction. In section III-B we present the explicit equations for the three-body cluster energy in terms of the state-dependent correlation functions and the effective interactions. Finally, in the last section we discuss our results, the various aspects of LOCV formalism, the three-body cluster energy and the uncertainties involved in the treatment of both the tensor and the channel dependence of the realistic interaction forces. In this respect we explain how it is possible to believe that

the LOCV method is capable of giving good agreement with the results obtained by the more complicated schemes in which the many-body contributions have been taken in to the account.

II. THE LOCV FORMALISM

We consider a trial many-body wave function of the following form to evaluate the Rayleigh-Ritz upper bound to the ground-state energy:

$$\psi_v = F\Phi \tag{1}$$

where Φ is a slater determinant of the plane waves of A independent nucleons,

$$\Phi = \mathcal{A}\prod_i \exp(i\vec{k}_i \cdot \vec{r}_i) \tag{2}$$

F is a A-body correlation operators which will be replaced by a Jastrow type i.e.

$$F(1\cdots A) = \mathcal{S}\prod_{i>j} f(ij) \tag{3}$$

and $\mathcal{A}$ and $\mathcal{S}$ are an anti-symmetrizing and a symmetrizing operators respectively. The cluster expansion energy is written as [13]:

$$E([f]) = \frac{1}{A}\frac{<\psi_v \mid H \mid \psi_v>}{<\psi_v \mid \psi_v>} = E_1 + E_2 + E_3 + \cdots \geq E_0 \tag{4}$$

where E_0 is the through ground-state energy. In the lowest order, we truncate above series after E_2.

The one body term E_1 is just the familiar Fermi gas kinetic energy i.e.

$$E_1 = \frac{3}{5}\frac{\hbar^2 {k_F}^2}{2m} \tag{5}$$

where $k_F = (\frac{3}{2}\pi^2\rho)^{1/3}$ and ρ, are the Fermi momentum and the density of uniform nuclear matter, respectively. The two-body energy E_2 is:

$$E_2 = \frac{1}{2A}\sum_{ij} <ij \mid \mathcal{V}(12) \mid ij>_a \tag{6}$$

and the "effective interaction operator" $\mathcal{V}(12)$ is given by the following equation:

$$\mathcal{V}(12) = -\frac{\hbar^2}{2m}[F(12), [\nabla_{12}^2, F(12)]] + F(12)V(12)F(12) \tag{7}$$

$< ij \mid O \mid ij >_a$ represent the antisymmetrized two-body matrix element taken with respect to the single-particle plane waves.

The two-body correlation operator $F(12)$ is defined as follows:

$$F(ij) = \sum_{\alpha,p=1}^{4} f_\alpha^{(p)} O_\alpha^{(p)}(ij) \tag{8}$$

$\alpha = \{J, L, S, T\}$ and operators $O_\alpha^{(p)}(ij)$ are written as:

$$O_\alpha^{p=1,4} = 1, (\frac{2}{3} + \frac{1}{6}S_{12}^I), (\frac{1}{3} - \frac{1}{6}S_{12}^I), S_{12}^{II} \tag{9}$$

where S_{12}^I is the familiar tensor operator and S_{12}^{II} is its analogous for the mixed N-Δ channel (in case of the Δ-Reid potential). The values of p is set to unity for $L = 0$ and the spin-triplet channels with $L \neq J \neq 1$. But for $L = J \pm 1$ it takes values of 2 and 3. Finally we have $L = 0$ channels which couple the ${}^1S_0$ channel to the ${}^5D_0$ channel (for the Δ-Reid potential) where we set $p = 1$ and 4. As in our previous works we also require that the correlation functions $f_\alpha^{(1)}$,$f_\alpha^{(2)}$ and $f_\alpha^{(3)}$ ($f_\alpha^{(4)}$) heal to the pauli function $f_P(r)$ (zero),

$$f_P(r) = [1 - [\frac{3}{2}l(k_F r)]^2]^{-1/2} \tag{10}$$

where ($J_J(x)$ are the spherical bessel functions)

$$l(x) = \frac{J_1(x)}{x} \tag{11}$$

The two-body nucleon-nucleon interaction $V(12)$ has the following form,

$$V(12) = \sum V^{(p)}(r_{12}) O_{12}^{(p)} \tag{12}$$

and they are taken from references [2,3].

The normalization constraint, $< \psi_v \mid \psi_v >= 1$, that we impose on the channel two-body correlation functions $f_\alpha^{(p)}$ as well as the coupled and uncoupled differential equations, coming from Euler-Lagrange equations are described in references [1,2].

III. THE THREE-BODY CLUSTER ENERGY

A. The state-averaged calculation

In general the three-body cluster term expression in the energy expectation value has the following form [13],

$$E_3 = E_3^{(2)} + E_3^{(3)} \tag{13}$$

where

$$E_3^{(2)} = -\frac{1}{A}\sum_{ijk} h_{ik} W_{ij} \tag{14}$$

$$E_3^{(3)} = \frac{1}{A3!}\sum_{ijk} Wijk \tag{15}$$

and h_{ik} , W_{ij} and W_{ijk} are defined as,

$$\begin{aligned} h_{ik} &= < ik \mid h(12) \mid ik >_a \\ W_{ij} &= < ij \mid \mathcal{V}(12) \mid ij >_a \\ W_{ijk} &= < ijk \mid \mathcal{V}(123) \mid ijk >_a \end{aligned} \tag{16}$$

The two-body effective operator $\mathcal{V}(12)$ is given in equation (7) and $(t(i) = -\frac{\hbar^2}{2m}\nabla_i)$

$$h(12) = F^\dagger(12)F(12) - 1 \tag{17}$$

$$\begin{aligned} \mathcal{V}(123) = \frac{1}{2}F_3^\dagger(123)\left[t(1)+t(2)+t(3)\ ,\ F_3(123)\right] \ + adj. \\ +(F_3^\dagger(123)V(12)F_3(123) - \mathcal{V}(12) + same\ for\ pairs\ 23\ and\ 13) \end{aligned} \tag{18}$$

Now by imposing the Jastrow approximation and ignoring the state and operator dependence of correlation functions, one can write the above three-body cluster energy $\bar{E}_3$ expressions in the following form :

$$\bar{E}_3 = \bar{E}_{3h}^{(2)} + \bar{E}_{3h}^{(3)} + \bar{E}_{3hh}^{(3)} + \bar{T}_{3hh}^{(3)} \tag{19}$$

where

$$\bar{E}_{3h}^{(2)} = -\frac{1}{A}\sum_{ijk} < ik \mid h(r_{13}) \mid ik >_a < ij \mid \mathcal{V}(r_{12}) \mid ij >_a \tag{20}$$

$$\bar{E}_{3h}^{(3)} = -\frac{1}{A}\sum_{ijk} < ijk \mid h(r_{13})\mathcal{V}(r_{12}) \mid ijk >_a \tag{21}$$

$$\bar{E}_{3hh}^{(3)} = \frac{1}{2A}\sum_{ijk} < ijk \mid h(r_{13})\mathcal{V}(r_{12})h(r_{23}) \mid ijk >_a \tag{22}$$

$$\bar{T}_{3hh}^{(3)} = \frac{1}{2A}\sum_{ijk} < ijk \mid \frac{\hbar^2}{2m} f^2(r_{12})\nabla_2 h(r_{12}) \cdot \nabla_2 h(r_{23}) \mid ijk >_a \tag{23}$$

and the hole function $h(r_{ij})$ is defined as,

$$h(r_{ij}) = f^2(r_{ij}) - 1 \tag{24}$$

$< ijk \mid O \mid ijk >_a$ represent the antisymmetrized three-body matrix element taken with respect to the single-particle plane waves.

Now, we define a state-averaged two-body correlation function [10] as

$$\bar{f}^2(x) = \frac{\sum_{\alpha,i}(2T+1)(2J+1)\frac{1}{2}[1-(-1)^{L+T+S}] f_\alpha^{(i)^2}(x) a_\alpha^{(i)^2}(x)}{\sum_{\alpha,i}(2T+1)(2J+1)\frac{1}{2}[1-(-1)^{L+T+S}] a_\alpha^{(i)^2}(x)} \tag{25}$$

and a state-averaged two-body effective interaction as

$$\bar{\mathcal{V}}(12) = \frac{\sum_{\alpha,i}(2T+1)(2J+1)\frac{1}{2}[1-(-1)^{L+T+S}] \mathcal{V}_\alpha^{(i)}(12) a_\alpha^{(i)^2}(x)}{\sum_{\alpha,i}(2T+1)(2J+1)\frac{1}{2}[1-(-1)^{L+T+S}] a_\alpha^{(i)^2}(x)} \tag{26}$$

where the $a_\alpha^i(x)$, etc. are

$$\begin{aligned}
a_\alpha^{(1)^2}(x) &= x^2 I_L(x) \\
a_\alpha^{(2)^2}(x) &= x^2(2J+1)^{-1}[(J+1)I_{J-1}(x) + JI_{J+1}(x)] \\
a_\alpha^{(3)^2}(x) &= x^2(2J+1)^{-1}[JI_{J-1}(x) + (J+1)I_{J+1}(x)] \\
a_\alpha^{(4)^2}(x) &= x^2 I_J(x)
\end{aligned} \tag{27}$$

with

$$I_L(x) = 48 \int_0^1 dz z^2 (1 - \frac{3}{2}z + \frac{1}{2}z^3) J_L^2(zx) \tag{28}$$

and

$$\bar{h}(ij) = \bar{f}^2(ij) - 1 \tag{29}$$

Finally after doing some algebra, we get the following three-body cluster terms in terms $\bar{h}(ij)$ and $\bar{\mathcal{V}}(ij)$ as,

$$\bar{E}_{3h}^{(2)} = \frac{\rho^3}{4A} \int d^3r_1 d^3r_2 d^3r_3 \; \bar{h}(r_{13})\, \bar{\mathcal{V}}(r_{12}) \\ \{4 + \frac{1}{4} l(k_F r_{23}) l(k_F r_{12}) l(k_F r_{13}) - (l^2(k_F r_{12}) + l^2(k_F r_{13}))\} \tag{30}$$

$$\bar{E}_{3h}^{(3)} = \frac{\rho^3}{4A} \int d^3r_1 d^3r_2 d^3r_3 \; \bar{h}(r_{13})\, \bar{\mathcal{V}}(r_{12}) \\ \{4 + \frac{2}{4} l(k_F r_{12}) l(k_F r_{23}) l(k_F r_{13}) - l^2(k_F r_{12}) - 2l^2(k_F r_{23})\} \tag{31}$$

$$\bar{E}_{3hh}^{(3)} = \frac{\rho^3}{8A} \int d^3r_1 d^3r_2 d^3r_3 \; \bar{h}(r_{13})\, \bar{h}(r_{23})\, \bar{\mathcal{V}}(r_{12}) \\ \{4 + \frac{2}{4} l(k_F r_{12}) l(k_F r_{23}) l(k_F r_{13}) - l^2(k_F r_{12}) - 2l^2(k_F r_{23})\} \tag{32}$$

$$\bar{T}_{3hh}^{(3)} = \frac{\rho^3}{8A} \int d^3r_1 d^3r_2 d^3r_3 (\frac{\hbar^2}{4m} \bar{f}^2(r_{13}) \nabla_2 \bar{h}(r_{12}) \cdot \nabla_2 \bar{h}(r_{23})) \\ \{4 + \frac{2}{4} l(k_F r_{12}) l(k_F r_{23}) l(k_F r_{13}) - l^2(k_F r_{12}) - 2l^2(k_F r_{23})\} \tag{33}$$

Then we can simply add the contribution of the three-body cluster energy, $\bar{E}_3$, to the two-body energy E_2.

In the figure 1 and 2 we plot the contributions of various quantities such as $\bar{E}_3$, $\bar{E}_{3h}^{(2)}$, $\bar{E}_{3h}^{(3)}$, $\bar{E}_{3h} = \bar{E}_{3h}^{(2)} + \bar{E}_{3h}^{(3)}$, $\bar{E}_{3hh}^{(3)}$, $\bar{T}_{3hh}^{(3)}$ and $\bar{E}_{3hh}^{(3)} + \bar{T}_{3hh}^{(3)}$ against density in nuclear matter for the Reid and the Δ-Reid potentials respectively. As we expect, it is seen that the main contribution comes from $\bar{E}_{3h}$. Since the other terms namely $\bar{E}_{3hh}$ and $\bar{T}_{3hh}$ are smaller than this term, i.e $\bar{E}_{3h}$ with order of $\bar{h}(ij)$. It is also seen that the density dependence of $\bar{E}_3$ is mainly due to $\bar{E}_{3h}$. On the other hand the size of $\bar{E}_{3hh}$ and $\bar{T}_{3hh}$ are roughly the same but in opposite sign beyond the nuclear matter density. So as we mention before, we can conclude that only $\bar{E}_{3h}$ is important in the three-body cluster energy.

B. The state-dependent calculation

Regarding our discussion in the section III-A , the main contribution of three-body cluster energy should come from $E_{3h}^{(2)}$ and $E_{3h}^{(3)}$. So in this section we explain how it is possible to write $E_{3h}^{(2)}$ and $E_{3h}^{(3)}$ in terms of $f_{\alpha}^{(i)}(r)$ and $\mathcal{V}_{\alpha}^{(i)}(r)$ and calculate them exactly i.e. without making state-averaged approximation.

By using the addition of angular momentum theorem [14], the partial wave expansion and various orthogonality relations, the explicit form of $E_{3h}^{(2)}$ can be written as following,

$$
\begin{aligned}
E_{3h}^{(2)} = \frac{-1}{A} \sum_{\{k_i k_j k_k\}} \sum_{LSJT} \sum_{L'J'S'T'} \int_0^\infty \int_0^\infty (xy)^2 \, dxdy \, ((-1)^{L+S+T} - 1)((-1)^{L'+S'+T'} - 1) \\
(\frac{(4\pi)^4}{\Omega^2}) h^{L'S'J'T'}(x) \mathcal{V}^{LSJT}(y) J_L^2(k_{ij}x) J_{L'}^2(k_{ik}y) \sum_{M_L M_{L'}} |Y_{LM_L}(\hat{k_{ij}})|^2 |Y_{L'M_{L'}}(\hat{k_{ik}})|^2 \\
\sum_{\{\sigma_i\sigma_j\sigma_k\}} \sum_{M_S M_{S'}} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & S \\ \sigma_i & \sigma_j & M_S \end{pmatrix}^2 \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & S' \\ \sigma_i & \sigma_k & M_{S'} \end{pmatrix}^2 \\
\sum_{\{\tau_i\tau_j\tau_k\}} \sum_{M_T M_{T'}} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & T \\ \tau_i & \tau_j & M_T \end{pmatrix}^2 \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & T' \\ \tau_i & \tau_k & M_{T'} \end{pmatrix}^2 \\
\sum_{M_J M_{J'}} \begin{pmatrix} L & S & J \\ M_L & M_S & M_J \end{pmatrix}^2 \begin{pmatrix} L' & S' & J' \\ M_{L'} & M_{S'} & M_{J'} \end{pmatrix}^2
\end{aligned}
\tag{34}
$$

where the large parentheses are the familiar Clebsch-Gordan coefficients and

$$
h^{LSJT}(x) =< LSJT \mid h(x) \mid LSJT > \qquad \mathcal{V}^{LSJT}(y) =< LSJT \mid \mathcal{V}(y) \mid LSJT >
$$

$$
\vec{x} = \vec{r}_1 - \vec{r}_3 \qquad \vec{y} = \vec{r}_1 - \vec{r}_2 \qquad \vec{k}_{ij} = \frac{1}{2}(\vec{k}_i - \vec{k}_j)
$$

$$
\vec{k}_{ik} = \frac{1}{2}(\vec{k}_i - \vec{k}_k) \qquad \vec{K}_{ik} = \vec{k}_i + \vec{k}_k \qquad \vec{K}_{ij} = \vec{k}_i + \vec{k}_j
$$

σ's and τ's stand for particles spin and isospin respectively. The following identity,

$$
\begin{aligned}
\sum_{l_1 l_2 m_1 m_2} j_{l_1}(qx) j_{l_1}(px) j_{l_2}(qy) j_{l_2}(py) Y_{l_1 m_1}(\hat{q}) Y_{l_1 m_1}^*(\hat{p}) Y_{l_2 m_2}(\hat{p}) Y_{l_2 m_2}^*(\hat{q}) = \\
\frac{1}{(4\pi)^4} \int d\Omega_x d\Omega_y \exp\left(i(\vec{x} - \vec{y}) \cdot (\vec{q} - \vec{p})\right)
\end{aligned}
\tag{35}
$$

is used to check the validity of equation (34). To see this matter clearly we assume that $\mathcal{V}$ and h are state-independent, then, it is possible to perform all of the summations and we show that one can reach to our previous equations for the state-independent three-body cluster energy (equation (30-33)).

In figures 3 and 4 we plot $E_{3h}^{(2)}$ by using the state-dependent correlation functions and effective potentials i.e. $f_\alpha^{(i)}(r)$ and $\mathcal{V}_\alpha^{(i)}(r)$ for Reid and Δ-Reid Potentials respectively. $E_{3h}^{(2)}(C)$ shows the same calculations but only with the central parts of $f_\alpha^{(i)}(r)$. $E_{3h}^{(2)}(NC)$ gives the estimate of non-central parts i.e. $E_{3h}^{(2)}$-$E_{3h}^{(2)}(C)$. From these figures, as one should expect, it can be concluded that the main contribution in $E_{3h}^{(2)}$ comes from the central parts of correlation functions and with very good approximation we can ignore the non-central parts of energy contributions in the three-body expressions (this is not a good approximation in the two-body level).

The exact calculation of $E_{3h}^{(3)}$ is very complicated, but according to the above argument we can take into the account only the central part of correlation functions and make it possible to have an estimate for this term as well i.e.

$$E_{3h}^{(3)}(c) = -\frac{1}{A}\sum_{ijk} < ijk \mid h(13)\mathcal{V}(12) \mid ijk >_a \tag{36}$$

We start by inserting a unit operator $\sum_{lmn} \mid lmn >< lmn \mid$ between the two-body operators $f(ij)$ and $\mathcal{V}(ij)$ in above equation.

$$E_{3h}^{(3)}(c) = \frac{1}{A}\sum_{ijk,lmn} < ijk \mid h(13) \mid lmn >< lmn \mid \mathcal{V}(12) \mid ijk >_a \tag{37}$$

Now by expanding the anti-symmetrized ket we get 9 terms which are very similar when we want to calculate them. We explain how it is possible to calculate one of these terms i.e.

$$\sum_{ijklmn} < ijk \mid h(13) \mid lmn >< lmn \mid \mathcal{V}(12) \mid ikj - kij >=$$

$$\sum_{ijkl} < ik \mid h(13) \mid lj >< lj \mid \mathcal{V}(12) \mid ik - ki >= \sum_{\{k_i k_j k_k k_l\}} \sum_{LSJT} \sum_{L'J'} \int_0^\infty \int_0^\infty dxdy$$

$$(xy)^2[T](\frac{(4\pi)^4(2\pi)^3}{\Omega^3})((-1)^{L+S+T}-1)h^{LSJT}(x)\mathcal{V}^{L'SJ'T}(y) \tag{38}$$

$$J_L(k_{ik}x)J_L(k_{lj}x)J_{L'}(k_{ik}y)J_{L'}(k_{lj}y)$$

$$\sum_{M_L M_{L'}} Y_{LM_L}(\hat{k_{ik}}) Y^*_{LM_L}(\hat{k_{lj}}) Y_{L'M_{L'}}(\hat{k_{lj}}) Y^*_{L'M_{L'}}(\hat{k_{ik}})$$

$$\sum_{M_J M_{\prime J} M_S} \begin{pmatrix} L & S & J \\ M_L & M_S & M_J \end{pmatrix}^2 \begin{pmatrix} L' & S & J' \\ M_{L'} & M_S & M_{J'} \end{pmatrix}^2 \delta((\vec{k}_i + \vec{k}_k) - (\vec{k}_l + \vec{k}_j))$$

In figures 5 and 6 we plot the various quantities such as $E^{(2)}_{3h}(C)$, $E^{(3)}_{3h}(C)$ and $E_{3h}(C)$=$E^{(2)}_{3h}(C)$ +$E^{(3)}_{3h}(C)$ for the Reid and the Δ-Reid potentials respectively. $\bar{E}^{(2)}_{3h}(C)$, $\bar{E}^{(3)}_{3h}(C)$ and $\bar{E}_{3h}(C)$ are the same terms but with the state-averaged approximation of section III-A.

Finally in figures 7 and 8 we plot the total contribution of $E_{3h}(C)$ by presenting the exact and the state-averaged calculation for the Reid and the Δ-Reid interactions. The size of $E_{3h}(C)$ is less than state-averaged $\bar{E}_{3h}(C)$ and this difference becomes sizable beyond the nuclear matter saturation density. This shows that at large densities the state-averaged approximation is no longer valid. Our final results for the binding energy of nuclear matter per nucleon are given in table 2 by adding the three-body cluster energies to our LOCV calculations.

IV. DISCUSSION

We have developed a method to make a very good estimate of the three-body cluster energy by using the state-dependent hole functions and effective interactions. We found that the size of the three-body cluster energy is much more smaller than our previous estimate using the state-averaged approximation. The smallness of the three-body cluster energy indicates that we are very close to the exact results at LOCV level and our cluster expansion series converges very rapidly. Our result with the Δ-Reid interaction is comparable with those calculations in which the three-body interactions [5-9] have been taken into the account such that they could reproduce the empirical properties of nuclear matter [5-9]. Presumably in the lowest order limit the cluster expansion for the exact two-body radial distribution function, $g(r)$, is automatically convergent and, provided that the lowest order energy has been efficiently minimized, the replacement of $g(r)$ by its lowest order approximation,

$$g^{(2)}(r) = \{1 - [\frac{3}{2} l(k_F r]^2\} f^2(r) \tag{39}$$

is then a very good approximation. In this case the integral constraint (normalization constraint),

$$\rho \int dr_{12} \, [1 - g(r_{12}] = 1 \tag{40}$$

plays a crucial role at all densities in forcing the correlation function to heal and in restricting the size of the wound in the correlated two-body wave function.

It is interesting to note that it may also be the inclusion of the tensor and spin-orbit forces and the incorporation of tensor correlation which allows us to deal accurately with the contributions to the energy of nuclear matter from all of the states of realistic potentials and to obtain such good agreement with far more ambitions and complicated calculations such as VHC.

Whether or not the above arguments are quantitatively correct, we still have to face that for $\rho \leq 0.3 fm^{-3}$ our nuclear matter results agree extremely well with those of VHC, which are hopefully reliable upper bounds on the energy up to this density and differ considerably from the accepted experimental equilibrium values. So we can conclude that only by taking into accounts the internal degrees of freedom of nucleons (such as $\Delta(1234)$) [2], the three-body forces [5-9] or the relativistic effect [16] it may be possible to remove this discrepancy between the theoretical calculation and the empirical prediction.

In this context we remark again that the LOCV method because of (1) its agreement with VHC which includes the many-body cluster contributions, (2) the smallness of three-body cluster energy and (3) its great simplicity, is a useful tool in the study of other properties of nuclear matter and finite nuclei.

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Figure captions :

Figure 1: Various terms of the three-body cluster energy (MeV) versus density (fm^{-3}) with the state-averaged correlation function and the effective interaction in nuclear matter according to the equations of section III-A for the Reid interaction.

Figure 2: Same as figure 1 but for the Δ-Reid interaction.

Figure 3: The plot of $E_{3h}^{(2)}$ (MeV) versus density (fm^{-3}) by doing an exact calculations with the Reid potential. $E_{3h}^{(2)}(C)$ and $E_{3h}^{(2)}(NC)$ are the central and non-central parts of this term, respectively.

Figure 4: Same as figure 3 but for the Δ-Reid potential.

Figure 5: Comparisons of $E_{3h}(C)$ (MeV), etc. with $\bar{E}_{3h}(C)$ etc. versus density (fm^{-3}) for the Reid potential

Figure 6: Same as figure 5 but for the Δ-Reid potential.

Figure 7: The plots of the state-averaged $\bar{E}_{3h}$ and its exact central contribution, $E_{3h}(C)$ (MeV) versus density (fm^{-3}) for the Reid interaction. $\bar{E}_{3h}(C)$ is the central part of state-averaged contribution

Figure 8: Same as figure 7 but for the Δ-Reid potential.

Table 1. The saturation energy, the saturation density and the incompressibility of nuclear matter for different potentials and the many-body methods (see different abbreviation and references on page 2).

Potential	Method	Author	$\rho_0(fm^{-3})$	E(MeV)	$\mathcal{K}$(MeV)
AV_{18}	LOCV	BM[6]	0.310	-18.46	302
AV_{14}	LOCV	BM[6]	0.290	-15.99	248
	VHC	WFF[8]	0.319	-15.60	205
	BB	DW[7]	0.280	-17.80	247
	BHF	BBB[9]	0.256	-18.26	-
UV_{14}	LOCV	BM[5]	0.366	-21.20	311
	VHC	CP[8]	0.349	-20.00	-
	VHC	WFF[8]	0.326	-17.10	243
UV_{14}+TNI	LOCV	BM[5]	0.170	-17.33	276
	VHC	WFF[8]	0.157	-16.60	261
	CBF	FFP[8]	0.163	-18.30	269
Δ-Reid	LOCV	MI[2]	0.258	-16.28	300
Reid	LOCV	OBI[1]	0.294	-22.83	340
	LOCV	MO[8]	0.230	-14.58	238
Empirical	-	-	0.170	-15.86	(200-300)

Table 2. The saturation energy and the saturation density of nuclear matter by adding the three-body cluster energy according to the text.

	Reid			Δ-Reid		
	E_2	$E_2+\bar{E}_3$	E_2+E_3	E_2	$E_2+\bar{E}_3$	E_2+E_3
$\rho_0(fm^{-3})$	0.294	0.216	0.295	0.258	0.208	0.257
E(MeV)	-22.83	-22.24	-23.362	-16.74	-15.597	-16.721